

Bounds on Effective Hamiltonians for Stabilizer Codes

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Abstract

This manuscript introduces various notions of k -locality of stabilizer codes inherited from the associated stabilizer groups. A choice of generators for the group leads to a Hamiltonian with the code in its groundspace, while a Hamiltonian holding the code in its groundspace might be called *effective* if its locality is less than that of a natural choice of generators (or any choice). This paper establishes some conditions under which effective Hamiltonians for stabilizer codes do not exist. Our results simplify in the cases of Calderbank-Shor-Steane stabilizer codes and topologically-ordered stabilizer codes arising from surface cellulations.

1 Introduction

A simple realization of a Hamiltonian can be achieved if the Hamiltonian is k -local for a small integer k . For adiabatic quantum computing, a Hamiltonian is of interest because its ground state reveals the answer to an interesting problem. We study in this paper the problem of determining lower bounds on the locality of Hamiltonians whose groundstate is a stabilizer code, and show that all such Hamiltonians must be at least as complicated as the underlying stabilizer group.

Consider a collection of n qubits evolving under a constant Hamiltonian H . Write $\mathcal{H}_1 = \mathbb{C}\{|0\rangle\} \oplus \mathbb{C}\{|1\rangle\}$ and the n -qubit Hilbert space as $\mathcal{H}_n = (\mathcal{H}_1)^{\otimes n} \cong \bigoplus_{j=0}^{2^n-1} \mathbb{C}\{|j\rangle\}$, so that we might view $H = \sum_{j,k=0}^{2^n-1} h_{jk} |j\rangle\langle k| \in \mathbb{C}^{2^n \times 2^n}$ as a Hermitian matrix. The notion of k -locality [KKR06] has been introduced to estimate how physically plausible such a Hamiltonian H might be. To describe this, let J be an n -long list of elements of $\{0, x, y, z\}$. For such an $J = j_1 j_2 \dots j_n$, we create an abbreviation $\sigma_{\otimes J} = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \dots \otimes \sigma_{j_n}$ for the appropriate tensor product of Pauli matrices. Let \mathcal{J} denote the set of all such indices J . We use $\mathcal{H}(2^n)$ to denote the vector space of Hermitian matrices. This sets notation for the equation

$$\mathcal{H}(2^n) = \bigoplus_{J \in \mathcal{J}} \mathbb{R} \{ \sigma_{\otimes J} \}. \quad (1)$$

Containment of the right-hand side follows since tensors of Hermitian matrices are Hermitian, while the equality follows from linear independence given that the Pauli-tensors are orthogonal in the matrix inner

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product $A \bullet B = \text{Trace}(AB^T)$ for $A, B \in \mathbb{C}^{2^n \times 2^n}$. Thus, H may also be written as

$$H = \sum_{J \in \mathcal{J}} t_J \sigma_{\otimes J}, \quad t_J \in \mathbb{R}. \quad (2)$$

If we use $\#J$ to denote the number of nonzero indices, then any summand $t_J \sigma_{\otimes J}$ of H denotes a $\#J$ -body interaction among the qubits. We say H is k -local when $k \geq \max\{\#J \mid t_J \neq 0\}$.

Much recent work in quantum complexity theory considers the ground states of k -local Hamiltonians. For example, an adiabatic quantum computer [FGLLP01, KKR06] must remain in the ground-state of a k -local Hamiltonian at all times. Early works on anyonic excitations of topologically ordered Hamiltonians [K03, FM01] used Hamiltonians whose addends were based on the local structure of some lattice. These were usually k -local for k small. Square lattices produce four local Hamiltonians while triangular lattices and their dual hexagonal lattices each produce six local Hamiltonians. Another recent topic considers realizing graph states as groundstates [NLDB07]. Realizing a graph state in this way is of interest since (i) the graph state is a *nondegenerate* groundstate which in principle could be obtained from the physical system by cooling, and (ii) any quantum circuit may be emulated using one-qubit rotations and measurements of the graph state [HDER06]. Thus realizing a Hamiltonian for a large enough graph state, cooling the system, and then applying local control and measurement is equivalent to universal quantum computation. Finally, recent work has considered a constrained family of Hamiltonians in order to produce new results on allowed groundstates [BVOT06].

Label \mathcal{P}_n as the group with elements $\{\pm \sigma_{\otimes J}\}_{J \in \mathcal{J}}$, and consider a subgroup $G \subset \mathcal{P}_n$. The stabilizer codespace of G is defined as

$$\mathcal{C}(G) \stackrel{\text{def}}{=} \left\{ |\psi\rangle \in \mathcal{H}_n \mid g|\psi\rangle = |\psi\rangle \quad \forall g \in G \right\}. \quad (3)$$

The codespace $\mathcal{C}(G)$ is nonzero [NC00] if and only if G is commutative. Now suppose we have a set S of $\pm \sigma_{\otimes J}$ that generate G and are at most k -local. Given commutativity, we may equally well think of the codespace as the ground eigenspace of the following Hamiltonian.

$$H_C = \sum_{\pm \sigma_{\otimes J} \in S} \mp \sigma_{\otimes J} \quad (4)$$

Even for rather small k , in fact even $k = 3$, engineering such a k -local Hamiltonian is challenging. Hence one wishes to find a Hamiltonian H_{eff} with the same groundstate eigenspace as H_C yet which is ℓ -local for $\ell < k$. Such a Hamiltonian H_{eff} is called an *effective Hamiltonian* for H_C . This paper provides conditions under which no such effective Hamiltonian exists.

Two applications result. First, consider that the Hamiltonian whose groundstate is a stabilizer code has created an energy-gap to leaving the code. This energy gap might be viewed as passive error correction, and our bounds on ℓ -locality of effective Hamiltonians become minimum expenses for obtaining such behavior. In particular, these results provide a quantitative argument that the four-local costs for toric codes [K03] and analogous codes for cellulated surfaces [FM01, BM07] are the best possible. A second application regards adiabatic computing, where attempts to drive down the required k -locality of adiabatic algorithms motivates the search for effective Hamiltonians [KKR06]. In that context, these arguments show that even effective Hamiltonians must be at least k -local for certain fixed k . However, such bounds are only on effective Hamiltonians which do not exploit ancillae. Of course, they still apply to systems with ancillae if the ancillae are included in a larger system.

The manuscript is organized as follows. Two notions of locality of a stabilizer subgroup of the Pauli group are introduced in §2, and each notion leads to a theorem constraining the inclusion of code spaces

into the groundstates of Hamiltonians which are excessively local. A perturbative variant in §3 shows that if the groundspace of an excessively local Hamiltonian is too near the stabilizer code, then the gap between the groundstate eigenvalue and the next distinct eigenvalue is pinched. Finally, we consider two examples in §4, namely Calderbank-Shor-Steane codes and stabilizer codes arising from cellulations of surfaces.

2 Stabilizer Codes as Exact Groundspaces

We write Pauli tensors as $\sigma_{\otimes J} = \sigma_{j_1} \otimes \sigma_{j_2} \otimes \cdots \otimes \sigma_{j_n}$ for $J = j_1 j_2 \dots j_n$ and each $j_k \in \{0, x, y, z\}$, where $\sigma_0 = \mathbf{1}$ and the other letters denote the usual Pauli matrices. For \mathcal{J} the set of all such indices J , the Pauli group \mathcal{P}_n is $\{\pm \sigma_{\otimes J} \mid J \in \mathcal{J}\}$. Thus $|\mathcal{P}_n| = 2 \cdot 4^n$, every nonidentity element $g \in \mathcal{P}_n$ has $g^2 = \mathbf{1}$, and all elements of \mathcal{P}_n commute or anticommute. For G a subgroup of \mathcal{P}_n , the stabilizer code of G is the subspace of \mathcal{H}_n which is the intersection of the $+1$ eigenspaces of all $g \in G$. It is known that the code space, say C , is nonzero if and only if G is commutative [NC00]. It is common to refer to G as a stabilizer group of C when (conversely) C is the intersection of the $+1$ eigenspaces of $g \in G$. Being less precise, a commutative subgroup $G \subseteq \mathcal{P}_n$ is a stabilizer group (of some nonzero C).

The discussion requires additional background on stabilizer codes. In particular, we highlight the following facts.

Lemma: [See [NC00, §10.5.1].] (i) Let $G \subseteq \mathcal{P}_n$ and $\Pi_G = (1/|G|) \sum_{g \in G} g$. Then for commutative G , Π_G is a projector onto the code space of G . Else $\Pi_G = \mathbf{0}$. (ii) If σ and $-\sigma$ are both in G , then the code space is trivial.

Proof: The first item is proven in the citation. For the second, the hypothesis requires $-\mathbf{1} = (\sigma)(-\sigma) \in G$. Thus $\text{Trace}(\Pi) = \text{Trace}(\mathbf{1} - \mathbf{1}) = 0$, since every element of \mathcal{P}_n other than $\pm \mathbf{1}$ is traceless. Since the projector Π is traceless, it is zero. Hence its target, the code space, is trivial. \square

Use $\text{wt}(g)$ for $g \in \mathcal{P}_n$ to denote the number of σ_x , σ_y , and σ_z factors of the tensor product. In particular, $\text{wt}(-\mathbf{1}) = 0$. Also, $\text{wt}(g_1 g_2) \leq \text{wt}(g_1) + \text{wt}(g_2)$, since any qubit whose tensor factors are $\mathbf{1}$ in g_1 and g_2 will have tensor factor $\mathbf{1}$ in their product. Finally, for $S \subset \mathcal{P}_n$, we use $\langle S \rangle$ to denote the subgroup generated by the elements of S . This is standard notation from abstract algebra, and we hope that context will make clear that it is not the Dirac notation for the expectation of an operator S .

Recall from Equation 1 that any Hamiltonian on n qubits may be written as a real linear combination of Pauli tensors. The Hamiltonian is k -local if the degree of no monomial summand exceeds k . This is a measure of complexity of the Hamiltonian and physical systems that realize it, in that k -local Hamiltonians require at most k -qubits to interact during any infinitesimal time.

This section presents two results which argue that Hamiltonians whose groundstate captures a stabilizer code must be at least as complicated as the underlying stabilizer group. The complication of Hamiltonians is measured in k -locality. On the other hand, two reasonable definitions of the k -locality of stabilizer group are considered in separate subsections. These two measures are motivated by earlier work [NLDB07] and so are denoted $\delta(G)$, a lower bound on the weight of $g \in G$, and $\eta(G)$, in principle an upper bound. We begin with $\delta(G)$.

2.1 Lower Bound Case

We now define a quantity $\delta(G)$ that may be viewed as a lower bound on the k -locality of a stabilizer group.

Definition: Let $G \subseteq \mathcal{P}_n$ be a subgroup. Then $\delta(G) = \min \{\text{wt}(g) \mid g \in G, g \neq \mathbf{1}\}$.

The next result implies that any Hamiltonian H whose groundstate is the stabilizer code must be at least $\delta(G)$ local. To see this, normalize so that H is traceless by subtracting the appropriate multiple of **1**. The groundspace of the traceless Hamiltonian is then a negative eigenspace.

Theorem 1: *Let G be a stabilizer group and let H be a traceless Hamiltonian on n -qubits which is k -local for $k < \delta(G)$. Let $\mathcal{V}_- \subset \mathcal{H}_n$ be the direct sum of eigenspaces of H corresponding to negative eigenvalues. Then the codespace of G is not contained within \mathcal{V}_- .*

Proof: Let $\{|\psi_j\rangle\}_{j=1}^L$ form a basis for the codespace. Recall Π from the proof of the Lemma:

$$\Pi = \sum_{j=1}^L |\psi_j\rangle\langle\psi_j| = (1/|G|) \sum_{g \in G} g.$$

While the second expression is an orthogonal decomposition of a projector, the third is a well known formula for a projector onto the code space [NC00, §10.5.1].

Recall the decomposition of the Hamiltonian H according to Equation 2 in the introduction.

$$H = \sum_{J \in \mathcal{J}} t_J \sigma_{\otimes J}, \quad t_J \in \mathbb{R}.$$

The traceless condition forces $t_{00\dots 0} = 0$, since for $J \neq 00\dots 0$ we have $\text{Trace}(\sigma_{\otimes J}) = \prod_{k=1}^n \text{Trace}(\sigma_{j_k}) = 0$. If some coefficient t_J is nonzero, then by hypothesis $\#J \leq k < \delta(G)$.

The estimate follows by considering $\text{Trace}(\sigma_{\otimes J} g)$ for $g \in G$ and $t_J \neq 0$. Then $\sigma_{\otimes J} g \neq \pm \mathbf{1} \in \mathcal{P}_n$, since the product has weight at least one. For g has weight at least $\delta(G)$ while $\sigma_{\otimes J}$ has weight at most $k < \delta(G)$, and $\sigma_{\otimes J} = \sigma_{\otimes J}^{-1}$, hence $\text{wt}(g) = \text{wt}(\sigma_{\otimes J}^{-1} \sigma_{\otimes J} g) \leq \text{wt}(\sigma_{\otimes J} g) + \text{wt}(\sigma_{\otimes J})$ or $\text{wt}(\sigma_{\otimes J} g) \geq \delta(G) - k$. Therefore $\text{Trace}(\sigma_{\otimes J} g) = 0$ since $\text{Trace}(h) = 0$ for any $h \in \mathcal{P}_n - \{\pm \mathbf{1}\}$. The right hand equality of the equation below follows.

$$\sum_{j=1}^L \langle \psi_j | H | \psi_j \rangle = \text{Trace}(\Pi H) = (1/|G|) \sum_{g \in G} \text{Trace}(gH) = 0. \quad (5)$$

Now if $\{|\psi_j\rangle\}_{j=1}^L \subseteq \mathcal{V}_-$, then each term at the far left of Equation 5 would be negative, leading to a contradiction. \square

How might one compute $\delta(G)$?

We now sketch how one might compute $\delta(G)$, using the stabilizer check matrix A of the stabilizer code. Thus $A = (A_X | A_Z) \in (\mathbb{F}_2)^{m \times n}$ corresponding to the choice of generators $\{g_j\}_{j=1}^m$, i.e. $G = \langle \{g_j\}_{j=1}^m \rangle$. A 1 in row k of column j of A_X corresponds to a factor of σ_x in qubit position k of generator g_j , and A_Z is similar. (See [NC00, eqn. (10.112)] or [HDER06, §2.2.3].) Since m is the number of generators for G and $g^2 = \mathbf{1}$ for any $g \in \mathcal{P}_n$, one way to calculate $\delta(G)$ would be to enumerate all 2^m products of generators. A possible optimization of this approach would be to delete generators until the set $\{g_j\}_{j=1}^m$ is *minimal*, i.e. until the number of rows of A is also its rank.

We present a different approach. Namely, suppose that a p -local tensor product $g = \pm \sigma_{\otimes J}$ is in G . The support of g will be given by $\text{supp}(g) = \{k \in \{1, 2, \dots, n\} \mid j_k \neq 0\}$, so that $|\text{supp}(g)| = p$. Then for v an indicator vector of which generators occur in the product for g , $v^T A = (w_X | w_Z)$ has w_X and w_Z zero outside entries indexed by S . Now label A_S as that matrix with the columns of A_X and A_Z corresponding to S replaced by zero entries. Then v^T is a left-null vector of A_S but not of A . On the other hand, any left-null vector of A , say w with $w^T A = 0$, must also satisfy $w^T A_S = 0$. Thus $\text{rank}(A_S) < \text{rank}(A)$.

Example: Consider $G = \langle X \otimes I \otimes Z, I \otimes Z \otimes X \rangle$, for which $\delta(G) = 2$. Taking a basis for A according to the generating set above yields the equation

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \quad (6)$$

Say $S = \{1, 3\}$, since the first row of A recovers the two-local $X \otimes I \otimes Z$ supported on these qubits. Then

$$A_S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \quad (7)$$

Thus the existence of this two-local element of G has caused $\text{rank}(A_S) < \text{rank}(A)$, which might also be inferred due to the left-null vector $v^T = (10)$.

Algorithm: Computing $\delta(G)$

For $k = 1, \dots, m$

For each $S \subseteq \{1, \dots, n\}$ with $\#S = k$ do:

Compute A_S by deleting columns of A corresponding to S .

If $\text{rank}(A_S) < \text{rank}(A)$ then

Return $\delta(G) = k$ and exit.

End for.

End for.

The algorithm above for computing $\delta(G)$ is polynomial in n , at least if $\delta(G) \in O(1)$. (Cf. [NLDB07].) Note however that it is not a polynomial time algorithm should $\delta(G) \in \Omega(n)$, since then the loop will loop over (a nonnegligible fraction of) the power set of $\{1, 2, \dots, n\}$.

2.2 Upper Bound Case

This section considers $\eta(G)$, which is an upper bound on the k -locality of G . However, we do not define $\eta(G)$ to be the maximum of weights of $g \in G$. For $|+\rangle^{\otimes n} = [2^{-1/2}(|0\rangle + |1\rangle)]^{\otimes n}$ spans the one-dimensional code space of the stabilizer group generated by the n Hermitian Pauli tensors $(\sigma_x)_j = \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \sigma_x \otimes \dots \otimes \mathbf{1}$ with a single Pauli- X on qubit j . Then G contains $\sigma_x^{\otimes n}$ of weight n , yet $|+\rangle^{\otimes n}$ is local. Thus to get a useful definition of an upper bound we resort to a minimax construction, taking the minimum over all generating sets of G of the maximum k -locality in a given set. The following definition (Cf. [NLDB07]) is equivalent to that minimax.

Definition: For $S \subseteq \mathcal{P}_n$, let $\langle S \rangle$ denote the subgroup generated by S . Let $G \subseteq \mathcal{P}_n$ be a stabilizer group with nontrivial codespace. Then $\eta(G)$ is the minimal v such that $\langle \{g \in G \mid \text{wt}(g) \leq v\} \rangle = G$.

Next we define a subgroup $G(\bar{b})$ and related notation.

Definition of $G(\bar{b})$: Let $G \subseteq \mathcal{P}_n$ be a commutative subgroup and $v < \eta(G)$. We label $G_v = \langle \{g \in G \mid \text{wt}(g) \leq v\} \rangle$. Fix a minimal generating set so that (i) $G_v = \langle \{g_j\}_{j=1}^v \rangle$ and (ii) $\text{wt}(g_j) \leq v$ for each j . Extend this to a minimal generating set so that $G = \langle \{g_j\}_{j=1}^t \rangle$, where the $g_j = \pm \sigma_{\otimes j}$ may be Pauli tensors of any degree if $j > v$. For a bitstring $\bar{b} \in \mathbb{F}_2^{n-v}$, we label a new subgroup of \mathcal{P}_n :

$$G(\bar{b}) = \langle \{g_1, g_2, \dots, g_v, (-1)^{b_{s+1}} g_{s+1}, \dots, (-1)^{b_t} g_t\} \rangle. \quad (8)$$

The generating set for $G(\bar{b})$ above is also minimal [NC00]. The dependence of $G(\bar{b})$ on η and on the (ordered) sequence of generators $\{g_j\}_{j=1}^t$ will be left implicit.

Say $v < \eta(G)$. We do not have a result which prohibits certain groundstate eigenspaces for v -local Hamiltonians as would be the case if $v < \delta(G)$. Yet there is a result which is similar to this, namely that the eigenspaces of v -local Hamiltonians may not distinguish G and other extensions $G(\bar{b})$. This might be of independent interest and will also imply a result similar to the previous one, except that the relevant energies of the traceless effective Hamiltonian must be positive rather than merely nonnegative.

Theorem 2: Let $v < \eta(G)$. Let Π_G and $\Pi_{G(\bar{b})}$ be projectors on the respective codespaces, where b is a bitstring and $G(\bar{b})$ is defined above. Then for any traceless v -local Hamiltonian H ,

$$\text{Trace}(\Pi_G H) = \text{Trace}(\Pi_{G(\bar{b})} H). \quad (9)$$

Proof: Let $\sigma \in \mathcal{P}_n$ such that $\text{wt}(\sigma) \leq v$. It suffices to show that $\text{Trace}(\Pi_G \sigma) = \text{Trace}(\Pi_{G(\bar{b})} \sigma)$. We prove this formula using a case study.

Case 1: Suppose either $\sigma \in G$ or $-\sigma \in G$ or both. Each such element is in G_v due to its weight, hence each such element is also an element of $G(\bar{b})$.

Since by hypothesis G has a nontrivial codespace, both σ and $-\sigma$ are not in G . (Else $-1 \in G$ and $\text{Trace}(\Pi_G) = 0$ contradiction.) As a remark, $\text{Trace}(\Pi_G \sigma) = \text{Trace}(\Pi_{G(\bar{b})} \sigma)$ nonetheless holds in this subcase as $0 = 0$.

Thus say $\sigma \in G$ with $-\sigma \notin G$ or vice-versa. Then each trace is $\pm 2^{t-n}$, since (i) $\text{Trace}(g_1 g_2) = 0$ whenever $g_1, g_2 \in \mathcal{P}_n$ and $g_1 \notin \{g_2, -g_2\}$ and (ii) the size of the minimal generating sets demand $2^{n-t} = \#G = \#G(\bar{b})$ [NC00].

Case 2: Suppose $\sigma \notin G$ and $-\sigma \notin G$. Then $\text{Trace}(\Pi_G \sigma) = 0$. It would suffice to show that $\sigma \notin G(\bar{b})$ and $-\sigma \notin G(\bar{b})$.

Assume by way of contradiction that $\sigma \in G(\bar{b})$. Then for a bit-string $\bar{c} = c_{s+1}c_{s+2}\dots c_t$, we have

$$\sigma = \prod_{j=1}^s g_j^{c_j} \prod_{j=s+1}^t \left((-1)^{b_j} g_j \right)^{c_j}. \quad (10)$$

Since $\prod_{j=s+1}^t (-1)^{b_j c_j} \in \{1, -1\}$, either $\sigma \in G$ or else $-\sigma \in G$. Contradiction. The case that $-\sigma \in G(\bar{b})$ is similar. \square

Corollary 3: Suppose G and $G(\bar{b})$ as in the Theorem. Suppose that H is a traceless, v -local Hamiltonian for $v < \eta(G)$. Partition $\mathcal{H}_n = \mathcal{V}_- \oplus \mathcal{V}_0 \oplus \mathcal{V}_+$ into positive, zero, and negative eigenspaces of H .

- If the codespace of G is contained within $\mathcal{V}_- \oplus \mathcal{V}_0$, then the codespace of $G(\bar{b})$ is contained within $\mathcal{V}_- \oplus \mathcal{V}_0$.
- Let $\mathcal{C}(\bar{b})$ denote the codespace of $G(\bar{b})$. If $\cup_{\bar{b}} \mathcal{C}(\bar{b})$ spans \mathcal{H}_n , then the codespace of G is not a ground eigenspace of any k -local Hamiltonian H .

How might one compute $\eta(G)$?

Recall the earlier algorithm to compute $\delta(G)$ using $A = (A_X | A_Z) \in \mathbb{F}_2^{m \times n}$. This section produces a similar algorithm for $\eta(G)$ using linear algebra. However, we first need some more notation. Namely, although the subset of k -local elements within G do not form a subgroup, those elements which only affect any collection of k -qubits do. The algorithm for $\eta(G)$ represents these subgroups as matrices and then uses algebra to decide whether their union generates G .

Definition: Recall the notation $\sigma_{\otimes J} = \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n}$ for $J = j_1 j_2 \dots j_n$ and $j_k \in \{0, x, y, z\}$, where $\sigma_0 = \mathbf{1}$ and the other sigmas denote the appropriate Pauli matrices. The *support* of $\pm \sigma_{\otimes J}$, $\text{supp}(\pm \sigma_{\otimes J})$, is $S = \{k \mid j_k \neq 0\} \subseteq \{1, 2, \dots, n\}$. Label the subgroup $\mathcal{P}_S = \{g \in \mathcal{P}_n \mid \text{supp}(g) \subseteq S\}$. Also set $G_S = G \cap \mathcal{P}_S$.

Henceforth, suppose G is fixed with *nontrivial codespace*, so that by the Lemma $g \in G$ demands $-g \notin G$. This creates a map from the row space of A to G . Indeed, since rows of A represent generators of G , the fact that the row vector $(b_1 b_2 \cdots b_n c_1 c_2 \dots c_n)$ lies within the row space implies $\pm(\sigma_x^{b_1} \otimes \sigma_x^{b_2} \otimes \cdots \otimes \sigma_x^{b_n})(\sigma_z^{c_1} \otimes \sigma_z^{c_2} \otimes \cdots \otimes \sigma_z^{c_n})$ is an element of G . Furthermore, although the $2n$ bitstring does not make clear the choice of sign, the Lemma asserts that it is unique. Now recall that A_S is the matrix A except that columns corresponding to $S \subseteq \{1, 2, \dots, n\}$ have been replaced with zero columns. As a consequence of the unique sign choice of the Lemma, any $2n$ -bit string in the rowspace supported on positions corresponding to S likewise determines an element of G_S . For \bar{S} the complement of S , such an element of the rowspace might be constructed by creating a left-null vector of $A_{\bar{S}}$.

Algorithm: Computing $\eta(G)$.

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for  $k = 1 : n$ 
   $A_k \in \mathbb{F}_2^{0 \times 2n}$ ;
  for  $S \subseteq \{1, 2, \dots, n\}$  with  $|S| = k$ 
    Compute  $N_S$ , a matrix whose rows span the left-null space of  $A$ .
    Compute  $B_S = N_S A_S$ , the matrix encoding  $G_S$ .
    Set  $A_k = \begin{pmatrix} A_k \\ B_S \end{pmatrix}$ .
    if  $\text{rank}(A_k) = \text{rank}(A)$ 
      return  $\eta(G) = k$ 
    End if.
  End for.
End for.
End for.
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3 Gap-Pinching when Approximating Stabilizer Codes

Our two earlier results limit those cases in which a stabilizer code lies within the groundstate of a Hamiltonian whose k -locality is less than some measure of the locality of the stabilizer group. The two measures of the group's locality were $\delta(G)$ and $\eta(G)$, where $\delta(G) \leq \eta(G)$. The result for $\delta(G)$ is stronger even though it applies to fewer Hamiltonians, in that it prohibits stabilizer codes within groundstates. In contrast, the more widely applicable result regarding $\eta(G)$ allows zero eigenspaces of traceless H to contribute and also requires some study of auxiliary stabilizer groups $G(\bar{b})$. Neither of these results was perturbative. We next present a result which limits those cases in which a stabilizer code is merely close to the groundstate of a Hamiltonian which is more local than the code. More precisely, we argue that the groundstate eigenspace of such a Hamiltonian lacks stability, in that the gap between the lowest two distinct eigenvalues is small when compared to the total energy of the system. Similar results regarding stabilizer codes for graph states are known [NLDB07].

The notation below will be fixed while discussing the perturbative result.

- Let $q = \dim_{\mathbb{C}} C(G)$, with $C(G)$ the stabilizer code of G . For a graph state, $q = 1$.

- Defining $v < \eta(G)$ and G_v as above, we let $r = \dim_{\mathbb{C}} C_v$ where $C_v = C(G_v)$. Similarly let $C = C(G)$. Since $G_v \subseteq G$, also $C_v \supseteq C$ and thus $r \geq q$.
- Consider H a v -local Hamiltonian with Π_H the projection onto its groundstate eigenspace.
- Π_G and Π_{G_v} are projections onto the appropriate stabilizer codespaces.
- The trace norm $\|\cdot\|_{\text{tr}}$ on Hermitian matrices is that norm induced by the inner product $H_1 \bullet H_2 = \text{Trace}(H_1 H_2^\dagger) = \text{Trace}(H_1 H_2)$.

In addition to the setup, we should note that one way to quantify the distance between the code of G and the groundstate of H is to compute the trace norm of the difference of the projectors onto each space.

Theorem: Let $G \subseteq \mathcal{P}_n$ have a code space of dimension $q > 0$. Let $v < \eta(G)$. Then any traceless v -local Hamiltonian H whose groundstate eigenspace is q -dimensional satisfies the following inequality on the trace norm distance between the projectors Π_G and Π_H onto the codespace of G and the groundstate eigenspace of H respectively.

$$\|\Pi_G - \Pi_H\|_{\text{tr}} \geq \frac{q}{\|\vec{E}\|_2} \left(\frac{E_0 + E_1 + E_2 + \cdots + E_{r-1}}{r} - E_0 \right). \quad (11)$$

Here, $E_0 \leq E_1 \leq \cdots \leq E_{2^n-1}$ is the eigenspectrum of H (with multiplicity) and $\|\vec{E}\|_2 = \text{Trace}(H^2)^{1/2} = (E_0^2 + E_1^2 + \cdots + E_{2^n-1}^2)^{1/2}$. Also, r denotes the dimension of the codespace of the group $G_v \subseteq G$ generated by v -local elements.

Proof: The first step is to check that due to the locality condition on H , we have $\text{Trace}(\Pi_G H) = (q/r)\text{Trace}(\Pi_{G_v} H)$. Since all elements of G and G_v that are at most v -local coincide, the following traces are equal.

$$\text{Trace} \left(H \sum_{g \in G} g \right) = \text{Trace} \left(H \sum_{g \in G_v} g \right). \quad (12)$$

The projectors should be normalized by $\#G$ and $\#G_v$ respectively. If m is the number of rows of a stabilizer check matrix for G arising from a minimal generating set and m_v is similar for G_v , then $\#G = 2^m$ and $\#G_v = 2^{m_v}$ [NC00]. Furthermore $q = 2^n/2^m$ and $r = 2^n/2^{m_v}$. Thus appropriately normalizing the above equation produces the desired equality.

Now let Π be any projection onto an r -dimensional space. Since $E_0 \leq E_1 \leq \cdots \leq E_{r-1}$ are the r least eigenvalues of H , we have the inequality

$$\text{Trace}(\Pi H) \geq (E_0 + E_1 + E_2 + \cdots + E_{r-1}). \quad (13)$$

Recall that the inner product associated to the trace norm has a Schwarz inequality. This is the final fact required for the following sequence of inequalities.

$$\begin{aligned} \|\vec{E}\|_2 \|\Pi_G - \Pi_H\|_{\text{tr}} &= \|H\|_{\text{tr}} \|\Pi_G - \Pi_H\|_{\text{tr}} \\ &\geq \text{Trace}((\Pi_G - \Pi_H)H) \\ &= (q/r)\text{Trace}(\Pi_{G_v} H) - \text{Trace}(\Pi_H H) \\ &\geq (q/r)(E_0 + E_1 + E_2 + \cdots + E_{r-1}) - qE_0. \end{aligned} \quad (14)$$

Appropriate manipulations of the inequality between the first and last expression of the sequence above produces the result. \square

Corollary: Let the traceless Hamiltonian H and code G satisfy all hypotheses of the theorem, including excessive locality of H as compared to $\eta(G)$. Label the spectral gap of H as $\Delta E = E_q - E_0$, recalling $E_0 = E_1 = E_2 = \dots = E_{q-1}$. The following estimate holds:

$$\| \Pi_G - \Pi_H \|_{\text{tr}} \geq q \|\vec{E}\|_2^{-2} ((r-q)/r) \Delta E. \quad (15)$$

In particular, if $\epsilon > \| \Pi_G - \Pi_H \|_{\text{tr}}$ and r and q are treated as constants, then the gap is pinched in the sense that $\Delta E \in O(\epsilon \|\vec{E}\|_2^2)$.

Proof: Notice that for $j \geq q$, $E_j \geq E_0 + \Delta E$. The term inside the parentheses of the Theorem is bounded below by a multiple of this gap (Cf. [NLDB07]):

$$\begin{aligned} \left(\frac{E_0 + E_1 + E_2 + \dots + E_{r-1}}{r} \right) - E_0 &\geq ((q/r)E_0 + ((r-q)/r)E_q) - E_0 \\ &= ((q-r)/r)E_0 + ((r-q)/r)E_q \\ &= ((r-q)/r)\Delta E. \end{aligned} \quad (16)$$

□

The pinching bound of the Corollary is weak in the following sense. (Cf. [NLDB07].) Effective Hamiltonians are used to approximate lower energy eigenstates while ignoring higher energy eigenstates. Thus the large total energy $\|\vec{E}\|_2^2$ is not a concern. On the other hand, the Corollary also argues that the higher energy eigenstates can not be (entirely) irrelevant to such approximations.

4 Examples

This section considers the computation of the quantities $\delta(G)$ and $\eta(G)$ used in the effective Hamiltonian bounds in special cases. We first note simplifications for a broad class of codes that includes CSS codes and also topological orders on surfaces. The topological order case requires further attention, in that answers should be computable using only the cellulation of the surface. The codes depend on the cellulation rather than the topology (i.e. genus) of the surface, and the same is true of $\delta(G)$ and $\eta(G)$.

4.1 Calderbank-Shor-Steane codes

Definition: Let $\mathcal{P}_{X,n} = \langle \{\sigma_{x,j}\}_{j=1}^n \cup \{-\sigma_{x,j}\}_{j=1}^n \rangle$ be the subgroup of \mathcal{P}_n containing Pauli tensors with only Pauli X factors, and let $\mathcal{P}_{Z,n}$ be similar. A stabilizer group G is XZ split if $G = \langle \{g_j\}_{j=1}^m \rangle$ where for each j either $g_j \in \mathcal{P}_{X,n}$ or $g_j \in \mathcal{P}_{Z,n}$. Perhaps upon reordering, this produces a block-diagonal stabilizer check matrix with blocks A_X and A_Z defined by the following equation:

$$A = \begin{pmatrix} A_X & \mathbf{0} \\ \mathbf{0} & A_Z \end{pmatrix}. \quad (17)$$

We also label $G_X = \mathcal{P}_{X,n} \cap G$ and $G_Z = \mathcal{P}_{Z,n} \cap G$.

All Calderbank-Shor-Steane codes [NC00, §10.4.2] [CS97, S96] are XZ -split. Indeed, suppose $\text{CSS}(C_1, C_2)$ is the code arising from classical codes C_1 and C_2 , where C_1 corrects bit-flips and C_2 phase-flips. Then for A_1 the parity check matrix of C_1 and A_2 the parity check matrix of the dual code C_2^\perp , we have a stabilizer check matrix $A = \text{diag}(A_1, A_2)$ for $\text{CSS}(C_1, C_2)$. The converse only holds in a technical sense¹. Next, we study $\delta(G)$ and $\eta(G)$ for XZ -split codes.

¹ Any XZ -split code might be associated to $\text{CSS}(C_1, C_2)$ for some classical codes C_1 and C_2 , yet the ratio of logical to encoding bits of these classical codes would be arbitrary. Thus we retain XZ -split as a separate concept.

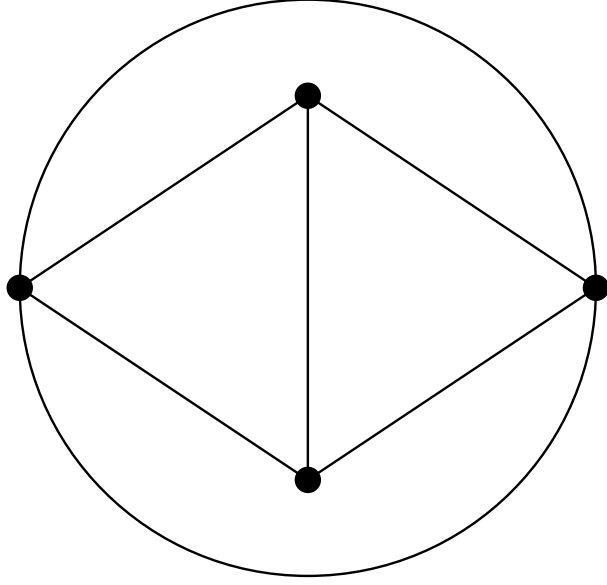


Figure 1: A counterexample to the conjecture that $\delta(G)$ is the minimum of the valences of the one-skeleta of the cellulation Γ and the dual cellulation Γ^* results as follows. Cellulate a disc as above. Cellulate a sphere with the top and bottom each copies of this disc. Then $\delta(G) = 2$, where the minimal boundary is the two-edge circle which bounds either disc. Yet the minimum of the valences is three.

Proposition: Suppose that G has nonzero code space. Label $G_X G_Z = \{g_x g_z \mid g_x \in G_X, g_z \in G_Z\}$.

- $(G \text{ is } XZ\text{-split}) \iff (G = G_X G_Z)$.
- If G is XZ split, then $\delta(G) = \min \{\delta(G_X), \delta(G_Z)\}$.
- If G is XZ split, then $\eta(G) = \max \{\eta(G_X), \eta(G_Z)\}$.

Proof: For the first item, since $G_X \subseteq G$ and $G_Z \subseteq G$, we must have $G_X G_Z = \{g_x g_z \mid g_x \in G_X, g_z \in G_Z\}$ within G . For the opposite containment, the generators guaranteed by the XZ -split condition show that $G = \langle G_X G_Z \rangle$. On the other hand, finite products of elements in $G_X G_Z$ lie in $G_X G_Z$, since G is commutative.

For the second item, the minimum is greater than $\delta(G)$ since $G_X \subseteq G$ and $G_Z \subseteq G$ imply $\delta(G_X) \leq \delta(G)$ and $\delta(G_Z) \leq \delta(G)$. On the other hand, let $g \in G$. Then $g = g_x g_z$ and $\text{wt}(g) \geq \max\{\text{wt}(g_x), \text{wt}(g_z)\}$ since any qubit on which either g_x or g_z has a nontrivial tensor factor will have a nontrivial factor in the product for g .

For the last item, let $G_X = \langle \{g_{x,j}\}_{j=1}^{m_x} \rangle$ and $G_Z = \langle \{g_{z,j}\}_{j=1}^{m_z} \rangle$ be generating sets chosen to be at most $\eta(G_X)$ local and $\eta(G_Z)$ local. Since $G = G_X G_Z$, we have $G = \langle \{g_{x,j}\}_{j=1}^{m_x} \cup \{g_{z,j}\}_{j=1}^{m_z} \rangle$. Thus $\eta(G) \leq \max\{\eta(G_X), \eta(G_Z)\}$.

On the other hand, assume by way of contradiction that $G = \langle \{g_j\}_{j=1}^m \rangle$ where every g_j has weight strictly less than $\max \{\eta(G_X), \eta(G_Z)\}$. Writing $g_j = g_{j,x} g_{j,z}$ produces generating sets $G_X = \langle \{g_{j,x}\}_{j=1}^m \rangle$ and $G_Z = \langle \{g_{j,z}\}_{j=1}^m \rangle$, each of which has weight less than the maximum. Contradiction. Thus we have also shown $\eta(G) \geq \max\{\eta(G_X), \eta(G_Z)\}$. \square

4.2 Topological orders from surface cellulations

This section considers $\delta(G)$ and also $\eta(G)$ in the case in which the stabilizer code G results from the cellulation of a surface without boundary [FM01, BM07]. We will not review the theory of cellulations or their duals, except to note that the dual cellulation associates a vertex to each face of the original and a face to each vertex (E.g. [H02]). The relevant definitions will imply that $\delta(G)$ is the number of edges in the smallest bounding chain in either the cellulation or its dual. We also provide a counterexample to the conjecture that $\delta(G)$ is the minimum of the valences of the one-skeleton and dual one-skeleton, although this is frequently the case in examples.

Let S be an oriented surface with no boundary, and let Γ be a two-complex which is a cellulation of S . Let $\mathcal{V}(\Gamma)$, $\mathcal{E}(\Gamma)$, and $\mathcal{F}(\Gamma)$ denote the vertices, edges, and faces of Γ respectively. We also suppose a dual cellulation Γ^* with bijections $\mathcal{V}(\Gamma^*) \leftrightarrow \mathcal{F}(\Gamma)$, $\mathcal{E}(\Gamma^*) \leftrightarrow \mathcal{E}(\Gamma)$, and $\mathcal{F}(\Gamma^*) \leftrightarrow \mathcal{V}(\Gamma)$ (E.g. [H02]).

Consider the quantum system which associates a qubit to each edge of Γ (or Γ^*). A well known topologically ordered stabilizer code has a code space whose dimension is $\dim_{\mathbb{F}_2} H_1(\Gamma, \mathbb{F}_2)$, where the latter is a cellular homology with bit coefficients [FM01, BM07]. To review this briefly, the generators are indexed by the unions of the faces and vertices of Γ . Let $q(e)$ denote the qubit of an edge $e \in \mathcal{E}(\Gamma)$ and $\sigma_{x,q}$ denote the Pauli tensor which is an identity except for a single σ_x factor on qubit q . Then the generator associated to a face $f \in \mathcal{F}$ is $\sigma_x^{\otimes f} = \prod_{e \in \partial f} \sigma_{x,q(e)}$. The generator associated to a vertex $v \in \mathcal{V}(\Gamma)$ may be defined in terms of the dual face $v^* \in \mathcal{F}(\Gamma^*)$. Namely, let $\sigma_z^{\otimes v} = \prod_{e^* \in \partial v^*} \sigma_{z,q(e^*)}$, or equivalently tensor over all qubits on edges incident on the vertex. The stabilizer code is then $G = \langle \{\sigma_x^{\otimes f}\}_{f \in \mathcal{F}(\Gamma)} \cup \{\sigma_z^{\otimes v}\}_{v \in \mathcal{V}(\Gamma)} \rangle$. Such a code is XZ -split. As an aside, the associated Hamiltonian $H = -\sum_{f \in \mathcal{F}(\Gamma)} \sigma_x^{\otimes f} - \sum_{v \in \mathcal{V}(\Gamma)} \sigma_z^{\otimes v}$ is of interest independent of its homologically structured degenerate groundstate, in that the excitations out of this groundstate are abelian anyons with $\mathbb{Z}/2\mathbb{Z}$ gauge [K03].

Before considering the topological order as an XZ -split stabilizer code, we set the following notation for homological boundary operators.

$$\begin{aligned} \partial_2 : \text{span}_{\mathbb{F}_2} \mathcal{F}(\Gamma) &\rightarrow \text{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma), \\ \partial_2^* : \text{span}_{\mathbb{F}_2} \mathcal{F}(\Gamma^*) &\rightarrow \text{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma^*). \end{aligned} \quad (18)$$

Consider matrices D_X and D_Z for ∂_2 and ∂_2^* respectively. Consider a column of D_X . It contains entries of 1 in \mathbb{F}_2 at precisely those positions corresponding to edges $e \in \mathcal{E}$ such that $e \in \partial_2 f$ for $f \in \mathcal{F}$ the column label. A similar comment applies to D_Z , so that the stabilizer check matrix of G has this form:

$$A = \begin{pmatrix} D_X^T & \mathbf{0} \\ \mathbf{0} & D_Z^T \end{pmatrix}. \quad (19)$$

Here, the superscript T denotes transpose. Also, we list face operators before vertex operators when forming the matrix, else an antidiagonal matrix results. Thus, in the special case of a topological order, it is possible to compute $\delta(G)$ and $\eta(G)$ using only homological inputs, namely the matrices of the appropriate boundary maps in the cellulation and cocellulation.

However, $\delta(G)$ and $\eta(G)$ clearly depend on the cellulation rather than the topology of the underlying surface. To emphasize that point, note that for $g \in G_x$ we may associate $|\text{supp}(g)|$ to the size of a boundary in $\text{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma)$, while a similar comment applies to g_z and $\text{span}_{\mathbb{F}_2} \mathcal{E}(\Gamma^*)$. Hence, $\delta(G)$ is the minimum of the smallest number of edges required to support a boundary in either Γ or Γ^* . Since one may always subdivide an edge, this is not a topological invariant.

It is tempting given the last paragraph to conjecture that $\delta(G)$ is the minimum of the valences of the one-skeleta of Γ and Γ^* , i.e. of the graphs which result by ignoring faces (two-cells) in either. In fact, this is incorrect. Figure 1 provide a counterexample, in that the boundary with the least number of edges in Γ does not bound a single face.

5 Conclusions

Since the locality of a Hermitian matrix might serve as a crude figure-of-merit for its experimental difficulty, theorists hope to find interactions which are both highly local and have robust stabilizer quantum codes as their groundstate eigenspaces (E.g. [DBB07]). This manuscript constrains such efforts by arguing that excessively local effective Hamiltonians either do not exist or else have undesirable properties. The three main results argue that (i) no effective Hamiltonian may be more local than the minimum locality of any element of the stabilizer group, (ii) effective Hamiltonians which do not allow for more nonlocal Pauli tensors in the stabilizer group must also be effective Hamiltonians for many other stabilizer groups, and (iii) approximating a stabilizer code using an excessively local Hamiltonian leads to gap pinching. Nonetheless, the technical statements given here might lead to new examples.

Speculating in a slightly broader context, there are two ways one might attempt to use k -local Hamiltonians to simulate $\ell > k$ -local systems of interest. One might (a) exploit crosstalk mediated by an ancilla or (b) pulse noncommuting Hamiltonians (absent ancilla). Our results do not account for clever use of ancilla. Indeed, ancillae have been used successfully to construct effective Hamiltonians in the adiabatic computing literature [KKR06]. On the other hand, our results argue that Hamiltonians whose addends are *noncommuting* Pauli tensors must nonetheless obey certain locality constraints on their groundspaces. Thus, the ancilla-based approach might be preferable.

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